# Birzeit University
# Mathematics Department
# Math 337

**Test1** **First Semester 2021/2022**

---

**QI: ) 12 points** disprove each of the following

**1)** If $G$ is a group and $H = \{x^2 : x \in G\}$, then $H$ is a subgroup of $G$

$A_4$ has 9 square elements that do not form a subgroup

or

In $S_3$, $H = \{\varepsilon, (12), (23), (13)\}$ which is not a subgroup

**2)** $S_n$ is a cyclic group for $n > 2$

$S_n$ is not an abelian group for $n > 2$

$Z(S_n) = \{\epsilon\}$ for $n \geq 3$.

**QII: 88)** Do the following

**1)** Let $G$ be an abelian group and $H = \{x^2 : x \in G\}$. Show that $H$ is a subgroup of $G$

see notes

First $H \neq \phi$ since $e = e^2 \in H$

Let $a, b \in H$ so there exist $x, y \in G$ such that $a = x^2, b = y^2$ and so $ab^{-1} = x^2y^{-2} = (xy^{-1})^2 \in H$

By one step test $H \leq G$

Or use two step

**2)** If $G$ is a group and $H$ a subgroup of $G$, $x \in G$ and $x^{-1}Hx = \{x^{-1}hx : h \in H\}$. Show that $x^{-1}Hx$ is a subgroup of $G$

First $xHx^{-1} \neq \phi$ since $e = xex^{-1} \in xHx^{-1}$

Let $a, b \in xHx^{-1}$ so there exist $h_1, h_2 \in H$ such that $a = xh_1x^{-1}, b = xh_2x^{-1}$ and so $ab = xh_1x^{-1}xh_2x^{-1} = xh_1xh_2x^{-1} \in xHx^{-1}$ since, $h_1h_2 \in H$ since $H \leq G$. Also if $a = xhx^{-1} \in xHx^{-1}$ then $a^{-1} = xh^{-1}x^{-1} \in xHx^{-1}$

By two step test $xHx^{-1} \leq G$

**3)** Find all elements of order 10 in $Z_{40}$

$1^k = k$ such that $gcd(10, k) = 1$

**4)** Show that $A_n$ is a subgroup of $S_n$

see notes

**5)** Let $G$ be a group and let $a, b \in G, ab = ba$ and the orders of $a, b$ are relatively prime. Prove that $|ab| = |a||b|$.

Show first if $(|a|, |b|) = 1$ then $< a > \bigcap < b >= \{e\}$.

Let $(|a|, |b|) = 1$ and $x \in< a > \bigcap < b >$ then $|x|$ divides both $|a|, |b|$ and since $(|a|, |b|) = 1$. So $|x| = 1$ and so $x = e$

Now, let $|a| = m, |b| = n, |ab| = k$, so $(ab)^{mn} = a^{mn}b^{mn} = e$ and so$|ab| \leq mn$. Also, $|ab| = k$ implies $(ab)^k = e$ and so $a^k b^k = e$. So $a^k = b^{-k} \in< a > \bigcap < b >= \{e\}$ and so $a^k = b^{-k} = e$, and so both $m, n$ divides $k$, and so $mn \leq k$

**6)** Let $G$ be a group such that $(ab)^2 = a^2b^2$ for all $a, b \in G$. Show that $G$ is abelian.

Let $a, b \in G$ then , $(ab)^2 = abab = a^2b^2$. By cancelation laws we get $ba = ab$. So $G$ is abelian.

**7)** Show that $Z(S_n) = \{\epsilon\}$ for $n \geq 3$.

Let $x, y$ be distinct and let $\alpha \in S_n$ such that $\alpha(x) = y$, if $\alpha(y) = x$, choose $\beta \in S_n$ such that $\beta(x) = y, \beta(y) = z \neq x$ and if $\alpha(y) = z \neq x$ then choose $\beta(xy)$

**8)** Find all generators of $Z_{30}$

generators are $1^k = k$ such that $gcd(30, k) = 1$

**9)** If $G$ is a group and $a \in G$ of order $n$ and $k$ divides $n$ then $|a^k| = \frac{n}{k}$

see notes

**10)** Let $G$ be a group, $a, b \in G$. Prove that $|aba^{-1}| = |b|$.

Let $|aba^{-1}| = n, |b| = m$. Show $m = n$

$|aba^{-1}| = n$, so $(aba^{-1})^n = e$ and so $ab^na^{-1} = e$ and so $b^n = a^{-1}a = e$ and $|b| = m \leq n$.

Also $|b| = m$ so $(aba^{-1})^m = ab^ma^{-1} = aea^{-1} = e$ and so $|aba^{-1}| = n \leq m$. So $m = n$

**11)** Let $\alpha = (1245)(23567)(1345) = (352)(467)$. Find $|\alpha| = gcd(3, 3) = 3$